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ON A BETA FUNCTION INEQUALITY II

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Abstract

In this paper, we extend an inequality of Alzer concerning the beta function for $[1, \infty) \times [1, \infty)$. Moreover, we show that this inequality is sharpening a result of Suryanarayana et al.. Some elementary inequalities of two real variables are proved.

1. Introduction

For $x > 0$, the classical gamma function Γ and the psi function or digamma function Ψ are defined as:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad \text{and} \quad \Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

Closely related to the gamma function is the beta function, which is the real function of two variables defined by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad x > 0, \ y > 0.$$

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A well-known equation connecting the beta and the gamma functions is

$$B(x,\ y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}. \tag{1.1}$$

For a proof of (1.1), see, for example, [9], where a good reference for these functions is also given. While in the recent past, several articles have appeared providing various inequalities for the gamma and polygamma functions, see [2, 3, 6, 7, 11, 12, 13, 15] and the references therein, only a few inequalities concerning the beta function can be found in the literature [4, 5, 10, 11, 14, 15, 16, 17]. Among the various kinds of inequalities concerning the beta function, we will select a special one first which will be considered in detail on $(0, 1] \times (0, 1]$.

Dragomir et al. [11, p.114, Theorem 3] established the relation

$$B(x,\ y) \le \frac{1}{xy} \quad \text{for} \quad 0 < x,\ y \le 1.$$

Recently, Alzer [5, p. 738, Theorem 3.1] obtained the following improved results for all $x,\ y \in (0, 1]$:

$$\frac{1}{xy}\left(1 - \alpha\frac{1 - x}{1 + x}\,\frac{1 - y}{1 + y}\right) \le B(x,\ y) \le \frac{1}{xy}\left(1 - \beta\frac{1 - x}{1 + x}\,\frac{1 - y}{1 + y}\right), \tag{1.2}$$

with the best possible constants $\alpha = 2/3\pi^2 - 4 = 2.57973\ldots$ and $\beta = 1$, respectively.

In [14, p. 338, Theorem], it was shown that the right hand side of (1.2) could be further strengthened, in fact, we have

$$\frac{1}{xy}(x + y - xy) \le B(x,\ y) \le \frac{1}{xy}\,\frac{x + y}{1 + xy}, \quad 0 < x,\ y \le 1. \tag{1.3}$$

We note that the left hand side of (1.2) and the left hand side of (1.3) are not comparable to each other. Now, we select and investigate some known beta function inequalities on $[1, \infty) \times [1, \infty)$. Cerone [10, p. 80, Theorem 7] presented the following estimation for $x > 1$ and $y > 1$:

$$\frac{1}{xy}\left(1 - \frac{x-1}{\sqrt{2x-1}}\frac{y-1}{\sqrt{2y-1}}\right) \leq B(x,\ y). \tag{1.4}$$

Let us define the function D by the following expression:

$$D(x,\ y) := \frac{1}{xy}\left(1 - \frac{(x-1)(y-1)}{\sqrt{(2x-1)(2y-1)}}\right) - \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Elementary computation gives $D(2,\ 3/2) = 1/90 > 0$. Moreover, it is easy to verify that $D(2,\ 2) = 0$ and $D(2,\ 3) = -1/36 < 0$. So, we conjecture that (1.4) could only be true for all real numbers $x \geq 2$ and $y \geq 2$.

Recently, Suryanarayana et al. [17, p. 3, Lemma 4.1] established the following inequalities:

$$\frac{2^{2-(x+y)}}{x+y-1} < B(x,\ y) < \frac{1-2^{2-(x+y)}}{x+y-1},\ x,\ y > 0 \text{ and } x+y \neq 1. \tag{1.5}$$

Let S denote the function

$$S(x,\ y) := \frac{2^{2-(x+y)}}{x+y-1} - \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Simple calculation reveals that $S(1/2,\ 1) = 2\sqrt{2} - \sqrt{2} = 0.8842\ldots > 0$, $S(1,\ 1) = 0$, and $S(1,\ 3/2) = (\sqrt{2} - 2)/3 = -0.1952\ldots < 0$. Therefore, it seems to us that the left hand side of (1.5) is only true for all $x,\ y \geq 1$.

Now, it is easy to show that for all $x,\ y \geq 4$, the left hand side of (1.4) is less sharp than the left hand side of (1.5), i.e.,

$$\frac{1}{xy}\left(1 - \frac{(x-1)(y-1)}{\sqrt{(2x-1)(2y-1)}}\right) < \frac{2^{2-(x+y)}}{x+y-1}, \tag{1.6}$$

since the left hand side of (1.6) is negative, whilst the right hand side is always positive.

The aim of this paper is to show that the right hand side of (1.2) remains true for all real $x,\ y \in [1,\ \infty)$. Moreover, a different lower bound for $B(x, y)$ for all $x, y \in [1, \infty)$ is also provided. More precisely, we show that the following inequalities hold for all real numbers $x, y \in [1, \infty)$:

$$\frac{x^{x-1}y^{y-1}}{(x+y)^{x+y-1}} < B(x, y) \leq \frac{2}{xy}\frac{x+y}{(x+1)(y+1)}. \tag{1.7}$$

2. Preliminary Lemmas

In order to establish the main theorem of this paper, we need some lemmas, which we present in this section. The lemmas deal with some useful formulae and inequalities concerning the Γ and Ψ functions. Moreover, we give an inequality of the logarithmic function in two variables. Furthermore, we offer a simple inequality involving the Γ function. In the first lemma, we collect some useful formulae, which can be found in [1, Chapter 6].

Lemma 2.1. *For all x, we have*

$$\Gamma(x+1) = x\Gamma(x). \tag{2.1}$$

Some special values are as follows:

$$\Gamma(1/2) = \sqrt{\pi}, \quad \Gamma(1) = \Gamma(2) = 1, \quad \Gamma(3/2) = \frac{1}{2}\sqrt{\pi}, \textit{ and } \Gamma(3) = 2.$$

Furthermore, we have

$$\Psi(x) = -\gamma - \frac{1}{x} + x\sum_{k=1}^{\infty}\frac{1}{k(x+k)}, \quad (x \neq 0, -1, -2, \ldots)$$

$$\Psi^{(m)}(x) = (-1)^{m+1}m!\sum_{i=0}^{\infty}\frac{1}{(x+i)^{m+1}}, \quad (x \neq 0, -1, -2, \ldots \textit{ and } m = 1, 2, \ldots) \tag{2.2}$$

$$\Psi^{(n)}(x+1) = \Psi^{(n)}(x) + (-1)^n\frac{n!}{x^{n+1}}, \quad n \geq 0. \tag{2.3}$$

Lemma 2.2. *The function*

$$h(t) := \Psi(t) + \frac{1}{t},$$

is strictly increasing on $(0, \infty)$.

Proof. This immediately follows from (2.2). □

Lemma 2.3. *Let* $0 < x,\ y \le 1$. *Then we have*

$$\frac{y^2}{x^2 + y^2 + 2xy - x^2y - xy^2} < \log\left(1 + \frac{y}{x}\right).$$

Proof. According to the elementary relation $2t/(2+t) < \log(1+t)$, $t > 0$ [16, p. 273, Theorem 3.6.18], it suffices to show that the following inequality holds:

$$\frac{y^2}{x^2 + y^2 + 2xy - x^2y - xy^2} - \frac{2y}{2x + y}$$

$$= -\frac{y(-2x^2 - 2xy + 2x^2y - y^2 + 2xy^2)}{(2x + y)(-x^2 - 2xy + x^2y - y^2 + xy^2)} < 0.$$

Since both the second factor of the numerator and the second factor of the denominator are negative, the lemma follows. □

The next result is due to Gordon [13, p. 861, Theorem 5].

Lemma 2.4. *For all* $t > 0$, *we have*

$$\log(t) - \frac{1}{2t} - \frac{1}{12t^2} < \Psi(t) < \log(t) - \frac{1}{2t} - \frac{1}{12(t + 1/14)^2}.$$

Lemma 2.5. *Let* $1 \le x,\ y < \infty$. *Then*

$$196y^3 + 420xy^2 + 224x^2y + 28x^2 + 28y^2 + 50xy + x + y$$

$$< 1176x^3y + 1176xy^3 + 2352x^2y^2,$$

holds.

Lemma 2.6. *For all* $t \geq 0$, *we have*

$$\frac{t^t}{(t+1)^{t+1}} < \frac{\Gamma(t+1)}{\Gamma(t+2)}. \tag{2.4}$$

Proof. From (2.1) after some simplifications, we obtain $t^t < (t+1)^t$, which proves (2.4). □

3. Main Result

Now, we are in the position to give the main result of this paper.

Theorem. *For all real numbers* $x,\ y \in [1, \infty)$, *we have*

$$\frac{x^{x-1}y^{y-1}}{(x+y)^{x+y-1}} < B(x,\ y) \leq \alpha \frac{1}{xy} \frac{x+y}{(x+1)(y+1)}, \tag{3.1}$$

with the best possible constant $\alpha = 2$. *Equality occurs in* (3.1) *if and only if* $x = y = 1$.

Proof. We first prove the left hand side of inequality (3.1). Let us define the following function f by:

$$f(x,\ y) := \log \frac{x^x y^y}{(x+y)^{x+y}} - \log \frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+1)}.$$

According to the symmetry in x and y, we may suppose that $1 \leq x \leq y$. Partial differentiation yields

$$\frac{\partial f(x,\ y)}{\partial x} = \log x - \Psi(x+1) + \Psi(x+y+1) - \log(x+y). \tag{3.2}$$

In order to show that $\partial f(x,\ y)/\partial x < 0$, we will give an upper bound for (3.2). Using (2.3) and Lemma 2.4, we get

$$\frac{\partial f(x,\ y)}{\partial x} = \log x - \Psi(x+1) + \Psi(x+y+1) - \log(x+y)$$

$$= \log x - \left[\Psi(x) + \frac{1}{x}\right] + \left[\Psi(x+y) + \frac{1}{x+y}\right] - \log(x+y)$$

$$< \frac{1}{2x} + \frac{1}{12x^2} - \frac{1}{x} + \left[-\frac{1}{2(x+y)} - \frac{1}{12(x+y+1/14)^2}\right] + \frac{1}{x+y}.$$

After some computations, we obtain

$$\frac{1}{2x} + \frac{1}{12x^2} - \frac{1}{x} + \left[-\frac{1}{2(x+y)} - \frac{1}{12(x+y+1/14)^2}\right] + \frac{1}{x+y}$$

$$= \frac{p_1(x,\ y)}{12x^2(x+y)(14x+14y+1)^2},$$

where we have

$$p_1(x,\ y) := 196y^3 + 420xy^2 + 224x^2y + 28x^2 + 28y^2 + 50xy + x + y$$

$$- \left(1176x^3y + 1176xy^3 + 2352x^2y^2\right).$$

On account of Lemma 2.5, we are lead to $p_1(x,\ y) < 0.$ Since $f(x,\ y)$ is strictly decreasing in x, we infer $f(x,\ y) \leq f(1,\ y).$ Applying Lemma 2.6 completes the proof of the left hand side of (3.1). To prove the right hand side of (3.1), let $1 \leq x \leq y.$ We wish to investigate the monotonicity property of function g defined by

$$g(x,\ y) := \log\Gamma(x+2) + \log\Gamma(y+2) - \log\Gamma(x+y) - \log(x+y) - \log\alpha.$$

We are going to show that $g(x,\ y) \leq 0.$ Partial differentiation gives

$$\frac{\partial g(x,\ y)}{\partial x} = \Psi(x+2) - \Psi(x+y) - \frac{1}{x+y}$$

$$= \Psi(x+1) + \frac{1}{x+1} - \Psi(x+y) - \frac{1}{x+y}.$$

We next claim – in view of Lemma 2.2 – that $\partial g(x, y)/\partial x < 0$, thus $g(x, y)$ is strictly decreasing in x, i.e.,

$$g(x, y) < g(1, y) = \log\Gamma(3) + \log\Gamma(y+2) - \log\Gamma(y+1) - \log(y+1) - \log\alpha$$

$$= \log\Gamma(3) - \log\alpha = \log 2 - \log\alpha \le 0,$$

which means $\alpha \ge 2$, hence the proof of the theorem is complete. □

In view of the right hand side of (1.2), it is clear that the right hand side of (3.1) is a simple extension of (1.2) for all $x, y \in [1, \infty)$.

4. Some Comparison Results

Now, we show that the right hand side of (3.1) improves the right hand side of (1.5).

Corollary 1. *For all real numbers* $x, y \ge 2$, *we have*

$$\frac{1}{xy}\frac{2(x+y)}{(x+1)(y+1)} < \frac{1-2^{2-(x+y)}}{x+y-1}. \tag{4.1}$$

Proof. Inequality (4.1) could be written as

$$4\,\frac{xy(x+1)(y+1)}{xy(x+1)(y+1) - 2(x+y)(x+y-1)} < 2^{x+y}.$$

Let the function f_1 be defined by

$$f_1(x, y) := \log\left[4\,\frac{xy(x+1)(y+1)}{xy(x+1)(y+1) - 2(x+y)(x+y-1)}\right] - \log\left(2^{x+y}\right).$$

Clearly, $f_1(2, 2) = \log(3/4) = -0.2876\ldots < 0$. Partial differentiation yields

$$\frac{\partial f_1(x, y)}{\partial x} = \frac{p_2(x, y)}{(x-1)x(x+1)(2x+2y+xy)},$$

where

$$p_2(x,\ y) := -(y\log 2 + \log 4)x^4 - (y\log 4)x^3 + (y\log 2 + \log 4 - 4)x^2$$

$$+ (\log 4 - 4)xy - 2y$$

$$< -(y\log 2 + \log 4)x^4 + (y\log 2 - y\log 4 + \log 4 - 4)x^2 - 2y < 0,$$

so, we conclude that $\partial f_1(x,\ y)/\partial x < 0,$ thus $f_1(x,\ y)$ is decreasing in x, i.e., $f_1(x,\ y) \le f_1(2,\ y) =: f_2(y).$ Taking into account that

$$f_2'(y) = \frac{(\log 2)y^2 - (\log 2)y + 1}{y(1-y)} < 0,$$

thus f_2 is strictly decreasing on $y \in [2, \infty),$ hence $f_2(y) \le f_2(2) = \log 3/4 < 0,$ which implies $f_1(x,\ y) \le f_1(2,\ y) = f_2(y) < 0,$ as desired. □

The second corollary deals with the comparison of the left hand side of (1.3) and the left hand side of (3.1) on $x,\ y \in (0,\ 1]$.

Corollary 2. *For all real* $x,\ y \in (0,\ 1],$ *we have*

$$\frac{x^{x-1}y^{y-1}}{(x+y)^{x+y-1}} \le \frac{1}{xy}(x + y - xy).$$

Proof. Let us define the function g as follows:

$$g(x,\ y) := \log\frac{x^{x-1}y^{y-1}}{(x+y)^{x+y-1}} - \log\frac{x+y-xy}{xy}.$$

There is no loss of generality in supposing $0 < x \le y \le 1.$ Building the partial derivative yields

$$\frac{\partial g(x,\ y)}{\partial x} = \frac{g_1(x,\ y)}{(x+y)(xy-x-y)}, \tag{4.2}$$

where

$$g_1(x,\ y) := -y^2 - (x^2 + 2xy + x^2y - y^2 + xy^2)\log x$$
$$+ (x^2 + 2xy - x^2y + y^2 - xy^2)\log(x + y).$$

Next, we show that $g_1(x,\ y) \geq 0.$ However, this is equivalent to

$$\frac{y^2}{x^2 + y^2 + 2xy - x^2y - xy^2} < \log\left(1 + \frac{y}{x}\right).$$

By virtue of Lemma 2.2, we conclude in view of (4.2) that $\partial g(x,\ y)/\partial x < 0,$ therefore $g(x, y)$ is strictly decreasing in x. Since $g(x, y) \leq \lim_{x\to 0} g(x, y) = 0,$ the proof of the corollary is complete. □

Remark. Finally, we mention that the left hand side of (1.5) and the left hand side of (3.1) are not comparable to each other on $x,\ y \in [1, \infty).$

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